9 Spherical Model of Discrimination of Self-Luminous and Surface Colors

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ABSTRACT

The metrical structure and dimensionality of color space were studied using estimates of large color differences. Multidimensional scaling of the data shows that the dimensionality of the space that provides a linear relationship between interpoint distances and chromatic differences is 4. However, color points do not completely fill in the four-dimensional space, but are located on a spherical surface. The perceived differences between colors are measured by Euclidean interpoint distances between points in the color space rather than by spherical distances. The phenomena of unique hue and color opponency were used to correlate the Cartesian axes with four neurophysiological channels of color vision. Three spherical angles at each point on the sphere correspond exactly to hue, saturation, and brightness of spectral lights. The colors of monochromatic lights are represented by a curve on this sphere. The subset of equibright colors is located on the spherical surface in a three-dimensional subspace. The same results were obtained by reanalyzing Indow and Kanazawa's (1960) data on color discrimination for Munsell colors. Functions are defined that relate Munsell color characteristics—hue, chroma, and value—with the three spherical angles of a color point in four-dimensional space.

INTRODUCTION

Constructing a uniform color space is a difficult problem that has yet to be solved. A traditional approach to this problem uses threshold measurement techniques that were developed primarily for application to the measurement of unidimensional sensory characteristics. Typically, discrimination of a single color characteristic is measured separately, and these measurements are then synthesized in a unified Euclidean model of color discrimination (Hurvich & Jameson, 1955; Judd & Wyszecki, 1963; Vos & Walraven, 1972; Wyszecki & Stiles, 1982). Yet this approach relies on assumptions about the dimensionality of color space that are often unjustified. Additional complications arise from the need to combine different kinds of experimental data in the framework of a unified model.

Quite a different approach to the study of color vision has been developed in the area of multidimensional scaling (MDS) (Indow, 1980; Izmailov, 1980; Izmailov, Sokolov, & Chernorizov, 1989; Shepard, 1962a, 1962b; Shepard & Carroll, 1966; Sokolov & Izmailov, 1988). This approach is based on the analysis of large, suprathreshold differences (or similarities) between colors. The method lets researchers investigate all color characteristics simultaneously. MDS lets one reconstruct spatial and metric structures of color difference without a prior assumptions (Kruskal, 1964; Shepard, 1962a, 1962b; Torgerson, 1958). One common restriction imposed in MDS in the analysis of color discrimination data is that a linear relationship holds between perceived color differences and interpoint distances in Euclidean space (Indow & Uchizono, 1960; Izmailov et al., 1989; Judd, 1967; Shepard & Carroll, 1966). This requirement of color space uniformity is the major criterion used to evaluate models (Judd & Wyszecki, 1963; Wyszecki & Stiles, 1982).

In earlier work, my colleagues and I considered a spherical model of uniform color space that was constructed using MDS analysis of large color differences (Izmailov, 1980; Izmailov & Sokolov, 1991; Izmailov et al., 1989; Sokolov & Izmailov, 1983, 1988). We tried to compare the spherical model obtained using colored light stimuli to that constructed by Indow and his colleagues on the basis of color differences between Munsell chip colors (Indow, 1980; Indow & Kanazawa, 1960). The general procedure for constructing a spherical model from data concerning the colors of equibright monochromatic lights is described in our earlier work (Izmailov & Sokolov, 1991).

METHODS

A detailed description of the apparatus and of the brightness-matching technique is presented in earlier papers (Izmailov, 1980; Izmailov et al., 1989). The apparatus, briefly, provided a visual display that was formed by an optical system comprising a visual photometer that included an objective, a photometric cube, and an eyepiece with Maxwellian view. The field of view comprised a circular test field of monochromatic light that subtended about 2° of visual angle that was surrounded by a dark annulus with outer diameter 6°.

Three subjects with normal color vision were asked to rate the color difference between successively presented pairs of lights using an integer ranging from 0 (the two stimuli in the pair were identical) to 9 (the two stimuli were maximally dissimilar). Maximum dissimilarity was not defined in the instructions to the subjects, who were free to use any number for any pair of nonidentical stimuli. There were 17 equibright color stimuli (16 monochromatic lights and 1 white light) paired in 136 combinations; each of the 136 combinations was presented 10 times in randomized order. Stimulus duration and interstimulus interval were 0.5 sec, and the interpair interval was 5 sec.

MDS—ANALYSIS OF LARGE COLOR DIFFERENCES

Estimates of the dissimilarity of each pair of light's colors were averaged across presentations and subjects and organized as a triangular matrix of color differences. A metric MDS procedure was applied to the matrix (Indow & Uchizono, 1960; Shepard & Carroll, 1966; Sokolov & Izmailov, 1983); it returned as solution a three-dimensional Euclidean space.

That the solution is three dimensional is interesting in light of the traditional view that only two dimensions are sufficient for describing equibright colors. This view is based mostly on color mixture data and on threshold discrimination data (Hurvich & Jameson, 1955; Judd & Wyszecki, 1963; Wyszecki & Stiles, 1982). Large color differences are supposed, by extension, to be described in a two-dimensional space also.

Let us first consider the formal foundation of the three-dimensional solution in terms of characteristic roots and coefficients of correlation. As can be seen in Table 9.1, not only the first and second dimensions but also the third dimension have large characteristic roots. The increase in the coefficient of correlation does

TABLE 9.1
Characteristic Roots and Coefficients of Correlations Describing the First Six Dimensions of Euclidean Space Obtained by MDS Analysis of Color Differences Between Equibright Light Stimuli (column a)[a] and Munsell Colors (column b) That Varied in Hue and Chroma[b]

	Characteristic Root		*Coefficient of Correlation*	
Dimension	*a*	*b*	*a*	*b*
1	14785	24978	0.782	0.776
2	9820	12556	0.977	0.966
3	2195	3068	0.993	0.974
4	1135	1127	0.993	0.976
5	698	540	0.996	0.976
6	498	394	0.996	0.977

[a]Indow and Uchizono (1960).
[b]Indow and Kanazawa (1960).

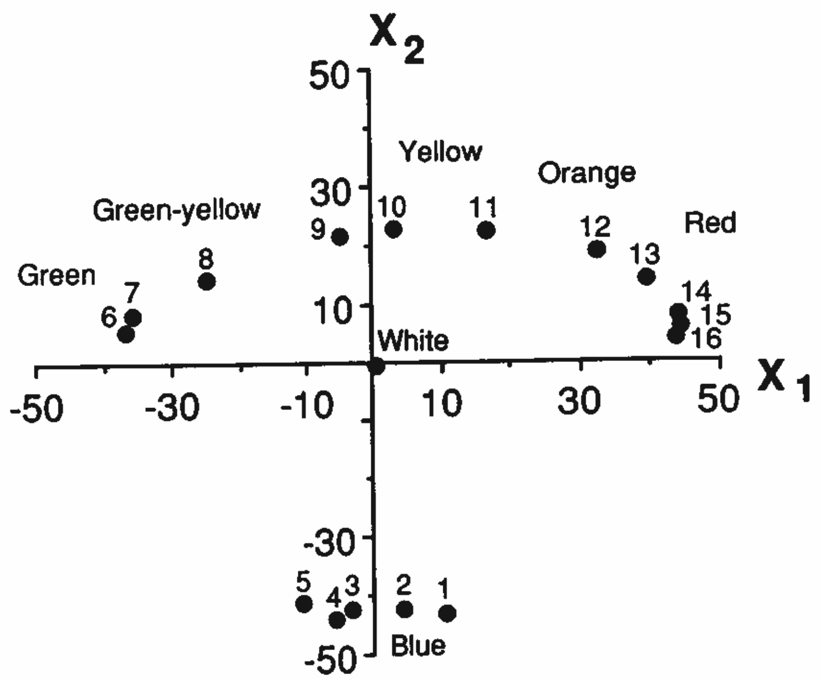

FIG. 9.1. Projections of the colors of monochromatic lights and a white light on the X_1X_2 plane formed of the first two dimensions. The white color is projected near the center of the X_1X_2 plane. Most saturated colors are projected far from the origin.

not end with the second dimension. While we could reject the third dimension if we had compelling reasons to do so, it will be shown that there are, on the contrary, strong reasons to consider the third dimension as a legitimate one in describing the data.

Figure 9.1 represents the traditional solution for spacing equibright colors in the X_1X_2 plane formed of the first two dimensions. However, there is a violation of color space uniformity, evident here as a considerable diminishing of linearity between perceived color differences and interpoint distances on the plane (Izmailov, 1980; Shepard, 1962a; 1962b; Shepard & Carroll, 1966). If the violation were due to measurement errors in the experimental data, then the distances between the color points and the X_1X_2 plane should be distributed randomly about zero. Figure 9.2 shows the projections of the color points onto the X_1X_3 plane of the same Euclidean space. It is evident from the data in Figures 9.1 and 9.2 that the spatial distribution of points along the third dimension is not random. There is a strict ordering in terms of color saturation. The most highly saturated colors (blue, green, and red), which are located far from the origin in the

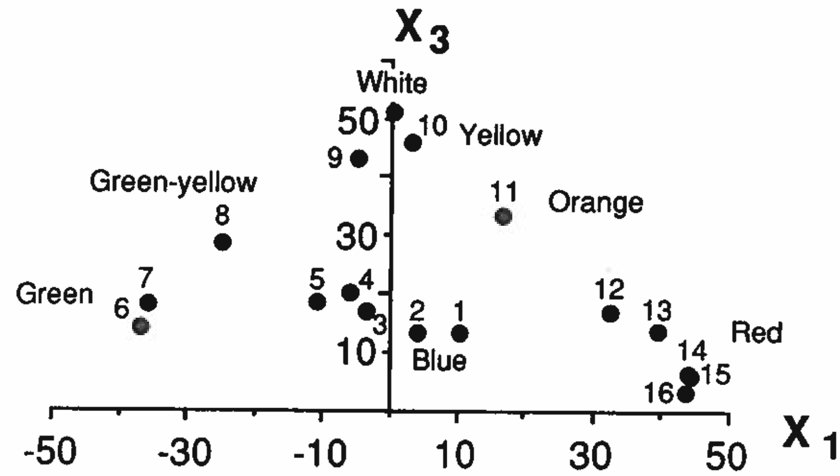

FIG. 9.2. Projection of the same colors on the X_2X_3 plane formed of the second and third dimensions. The white color point has the greatest height on this plane, while saturated colors have lesser heights.

chromatic plane X_1X_2, have small X_3 coordinates. The less saturated colors (green-yellow, yellow, and orange) are located closer to the origin of the X_1X_2 plane and have larger X_3 coordinates. The color with the greatest X_3 coordinate is white, which has small X_1 and X_2 values.

In other words, change in saturation corresponds to a special trajectory in the vertical plane of three-dimensional color space. Just as hue is characterized by azimuthal angle on the chromatic plane, saturation is described by vertical angle in three-dimensional space. The data show clearly that a third dimension is important for understanding color discrimination, particularly for describing changes in color saturation.

SPHERICAL PROPERTIES OF COLOR SPACE

An essential peculiarity of the obtained spatial structure is the following: the color points do not fill the three-dimensional space uniformly, rather they form a non-Euclidean surface of constant positive curvature within the three-dimensional space—namely a spherical surface. In order to prove this fact, it is necessary and sufficient to show that a geometric center can be found for a given configuration of color points.

Theoretically, the center must be equidistant from all color points. Yet the data are not free of error, so that one determines the center by finding the point for which the variance of the distances to all color points is minimized. I used an iterative procedure to minimize the variance of the radial distances by shifting the center point. A coefficient of variation is computed by taking the ratio of the standard deviation and the mean radius.

The goodness of fit to the spherical model is estimated in terms of both the coefficient of variance for radial distances and the coefficient of correlation between interpoint distances and initial color differences. A decrease in the size of the first coefficient, while maintaining the magnitude of the second one, represents a better fit. Values obtained from the given data are 0.078 and 0.995, respectively.

COLOR SPACE ROTATION

The interpoint distances, on which the resulting scaling solution is based, are invariant to all possible rotations. As a result, further motivation is needed for the particular directions that have been chosen for the cartesian axes of the solution space. In the spherical model of color discrimination, every axis is interpreted as opponent and so must be related to the color points in a certain way.

The well-known phenomenon of unique hue was used to orient the axes in the space. There are three hues in the spectrum seen when presenting monochromatic lights with wavelengths about 470 nm, 500 nm, and 575 nm and a further nonspectral hue (seen by mixing lights with dominant wavelengths 440 nm and 675 nm) that are considered as perceptually pure colors, namely "blue," "green," "yellow," and "red," respectively. Each such unique hue characterizes a single color characteristic in color-opponent theory (Hurvich & Jameson, 1955).

One axis in the solution space was chosen to be a red-green axis and was oriented to pass through the green point corresponding to wavelength 500 nm. Another axis was chosen to be a blue-yellow axis and was oriented to pass through the color points corresponding to wavelengths 470 nm and 580 nm. To obtain the frame of reference that represents the opponent properties of color vision in such a way, one rotates the initial configuration of points to reach as precise a localization of unique hue colors on the corresponding axes as possible.

The color-opponent spectral sensitivity functions that are derived from the final configuration are shown in Figure 9.3, and their similarity to corresponding functions, found empirically by using a hue cancellation technique (Hurvich & Jameson, 1955), helps to verify the solution.

NORMALIZATION OF THE COLOR SPHERE

A sphere has a constant radius for every point on its surface, speaking theoretically, but noise in the data cause radius values to fluctuate, and as a result the color sphere has a nonzero width which corresponds to the coefficient of variation (see Table 9.2). Since I do not deal with the interpretation of the radius

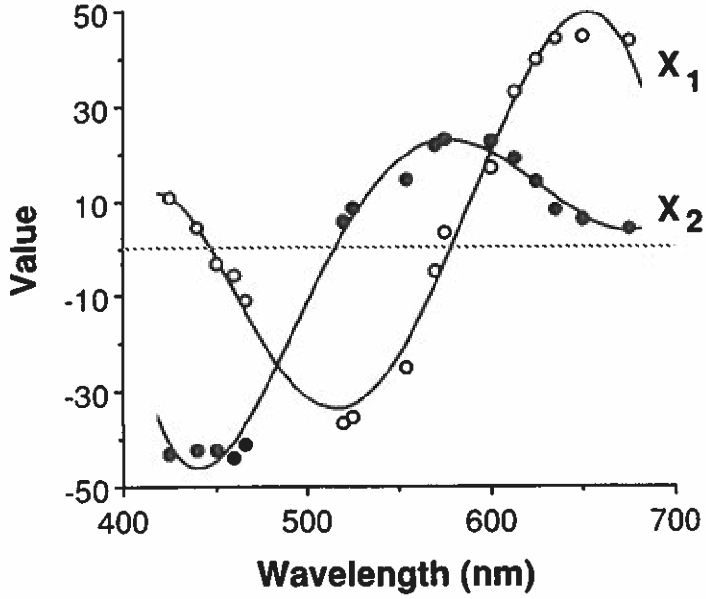

FIG. 9.3. The red-green (X_1) and blue-yellow (X_2) opponent functions derived from the spherical model of color discrimination. The curves are polynomials of degree 5 that fit the data points best.

TABLE 9.2
Coefficients of Variation Describing Sphericity of the Color Spaces Obtained by MDS Analysis of Color Differences Between Munsell Colors for the Six Original Paired Asymmetric Matrices (column a), for the Three Matrices Found by Averaging the Two Matrices for Each Subject (column b), the Two Matrices Found by Averaging Two Subjects' Matrices (1, 2, 3, 4 and 3, 4, 5, 6) (column c), and the Matrix Found by Averaging the Six Matrices of All Three Subjects (column d)

Matrices of Color Differences	*Coefficients of Variance (%)*			
	a	*b*	*c*	*d*
1	36.0	32.0		
2	34.0		29.0	
3	22.4	18.7		5.3
4	16.4		10.0	
5	16.3	12.7		
6	17.4			

values here, the sphere is reduced to a sphere of radius 1 by eliminating the fluctuations:

$$x_{1i}^2 + x_{2i}^2 + x_{3i}^2 = 1, \quad (9.1)$$

where

$$x_{ki} = X_{ki}/R_i, \qquad k = 1, 2, 3, \quad (9.2)$$

in which X_{ki} are the original coordinates and R_i is the radius of the ith color point.

ANALYSIS OF THE COLOR SPHERE

The presented model is a two-dimensional spherical surface in a three-dimensional Euclidean space. Each point on the surface corresponds to a particular color. Points that lie off of the spherical surface do not correspond to colors.

The colors of monochromatic lights are situated along a curvilinear path closed by purple colors. The pole of the sphere represents a "pure" white color. The azimuthal angle of a point codes its hue, while its vertical angle or elevation codes its saturation. The perceived difference between a pair of colors is determined by the central angle of the smaller of the two arcs determined by the great circle that passes through the two points. But quantitatively the difference is measured in terms of the line segment that joins the same points. This means that the structure of perceived color differences can be described in terms of a Euclidean metric:

$$d_{ij}^2 = \sum_{k=1}^{n} (X_{ki} - X_{kj})^2, \quad (9.3)$$

where d_{ij} is the distance between the ith and jth color points and n is the number of cartesian axes. This description holds true provided that each color point lies on the same spherical surface; i.e.,

$$S_i^2 = \sum_{k=1}^{n} X_{ki}^2 = \text{constant}. \quad (9.4)$$

THE BRIGHTNESS DIMENSION

For equibright colors, the number n of Cartesian axes Equation 9.3 is equal to 3. One possible way to visualize a new, second set of equibright colors at another level of luminance is as a concentric sphere of different radius. Yet another level of brightness would describe a third concentric sphere, etc.

We so arrive at the idea of Schrödinger (1920) and of Vos and Walraven (1972) that colors, varying in their chromatic characteristics as well as in brightness, fill an entire globe in three-dimensional Euclidean space.

Our experiments reject this idea.

What happens, in fact, when we add the dimension of brightness to the chromatic characteristics of the colors, is that we transform the spherical surface in three-dimensional Euclidean space to another spherical surface, this time in four-dimensional space.

Adding brightness variations increases the number of spherical axes. In addition to the two angles that are needed to characterize the hue and saturation of a color point, a third spherical angle is required to characterize its brightness.

This model has been confirmed by several independent experiments by my colleagues and me (Izmailov et al., 1989; Sokolov & Izmailov, 1983, 1988). Here I shall illustrate the spherical model of color space by applying it to Professor Tarow Indow's data on discrimination of Munsell colors.

REANALYSIS OF INDOW'S DATA

Our first step was to analyze the data of Indow and Uchizono (1960). Estimates of pair differences between 21 Munsell colors that varied in hue and chroma only were presented as a triangular matrix for one of four subjects.

Using MDS procedures, the coordinates of color points in 21-dimensional Euclidean space were obtained. Characteristic roots and coefficients of correlation were calculated for solution dimensionalities. As can be seen from Table 9.1 (columns marked b), the characteristic values stress the importance of the first and second dimensions. However, the third dimension also gives a prominent, additional contribution to the structure of the color points.

Goodness of fit to the spherical model of color vision was estimated for the Indow and Uchizono data as well as for the original color discrimination data. For points in three-dimensional space, the smallest coefficient of variance for radial distances of color points from the origin ("thickness" of the spherical layer) is 0.11 with the coefficient of correlation equal to 0.974.

Our second step was to apply this approach to Indow and Kanazawa's (1960) data. In this work, estimates of pair differences between 24 colors that varied in hue, chroma, and value were presented as pairs of asymmetric triangular matrices, one pair per subject. Each pair of matrices represented the data from a single subject; paired matrices differed in the order of stimuli. The first matrix represented estimates between i and j stimuli, while the second matrix represented estimates between j and i stimuli.

We used MDS to determine the coordinates of color points in four-dimensional space for the data of three subjects. For each of the subjects, we analyzed both the original pair of asymmetric matrices and also the matrices

TABLE 9.3
Coefficients of Correlation Describing Four-Dimensional Euclidean Space Obtained by MDS Analysis of Color Differences Between Munsell Colors for Separate Matrices (column a), Averaging for Each Subject (column b), Averaging for Two Subjects (column c), and Averaging for All Three Subjects (column d)

Matrices of Color Differences	*Coefficients of Correlation*			
	a	*b*	*c*	*d*
1	0.879	0.950		
2	0.896		0.971	
3	0.914	0.958		0.982
4	0.922		0.977	
5	0.923	0.964		
6	0.912			

found by averaging these. The goodness of fit by the spherical model of color vision to Indow and Kanazawa's data was determined. For the six original asymmetric matrices, the minimal coefficients of variance for radial distances ("thickness" of spherical layer for color points) varied from 16.3% to 36.0% (Table 9.2, column a), while the coefficients of correlation varied from 0.814 to 0.899 (Table 9.3, column a).

These results are bad from the point of view of one wishing to construct a spherical model. However, the data averaged for each subject give, in all cases, smaller coefficients of variation and larger coefficients of correlation (Table 9.2, column b and Table 9.3, column b). I suggest that the poor solutions obtained with the original pair of asymmetric matrices result from random errors in the initial estimates, and that averaging leads to better fits. In connection with this suggestion, we averaged data for two subjects (using two of the three possible groups of four matrices) and then for three subjects (all six matrices) and analyzed the resulting matrices the same way. Results are given in Tables 9.2 and 9.3 in columns c and d. The results show that averaging four matrices improves noticeably the characteristics of sphericity. Averaging all six matrices leads to a coefficient of variation equal to 5.3% and a coefficient of correlation equal to 0.982. These results agree with our work on the colors of monochromatic lights in showing a high degree of sphericity.

INTERCONNECTIONS BETWEEN MUNSELL AND SPHERICAL COLOR COORDINATES

I calculated the spherical coordinates representing hue, saturation and lightness of the 24 Munsell colors, using Equation 9.2 to normalize the four Cartesian

TABLE 9.4
Four Cartesian and Three Spherical Coordinates (in radians) of 24 Munsell Colors in Terms of the Spherical Model of Color Discrimination (Column R Lists the Radial Distance of Each Color Point from the Origin)

	Cartesian Coordinates					*Spherical Coordinates*		
Color	X_1	X_2	X_3	X_4	R	a_1	a_2	a_3
1	4.06	0.98	6.71	4.58	9.1	6.03	0.47	0.60
2	3.20	0.81	8.52	4.42	10.0	6.05	0.33	0.48
3	−1.46	2.93	8.47	4.53	10.0	4.23	0.34	0.49
4	−1.24	1.45	9.08	4.21	10.0	4.04	0.19	0.43
5	−5.50	0.23	7.37	4.33	10.0	3.18	0.57	0.53
6	−3.99	−0.16	7.86	3.61	9.6	3.09	0.44	0.43
7	−2.06	−4.27	7.16	4.66	9.8	2.02	0.51	0.58
8	−1.58	−3.13	8.24	4.38	9.9	2.05	0.36	0.49
9	3.09	−3.98	7.49	5.18	10.4	0.90	0.50	0.61
10	2.22	−3.33	8.60	4.63	10.5	0.96	0.38	0.49
11	2.92	0.98	6.40	5.26	8.9	5.92	0.35	0.69
12	1.93	−0.06	7.23	6.18	9.7	0.05	0.20	0.71
13	1.90	4.45	6.81	6.79	10.8	5.11	0.48	0.78
14	0.01	2.67	7.24	6.61	10.2	5.04	0.28	0.74
15	−4.45	0.66	6.85	6.77	10.6	3.30	0.45	0.78
16	−2.79	0.12	7.30	6.08	9.9	3.18	0.28	0.70
17	−3.03	−0.57	5.96	6.16	9.2	2.97	0.35	0.80
18	−2.45	−0.13	7.20	6.13	9.8	3.10	0.26	0.70
19	−3.13	−2.89	5.47	6.90	9.8	2.39	0.45	0.90
20	−2.21	−2.11	6.34	6.27	9.4	2.38	0.32	0.78
21	−1.59	−3.11	7.24	7.19	10.8	2.05	0.33	0.78
22	−1.27	−2.27	7.28	6.77	10.3	2.10	0.26	0.75
23	3.30	−0.17	6.27	6.82	9.8	0.06	0.35	0.82
24	2.32	−0.27	7.60	7.23	10.7	0.15	0.21	0.76

coordinates. Spherical coordinate values are listed in Table 9.4 in units of radians.

Hue. The hue of a color point is determined by its azimuthal angle in the chromatic plane x_1x_2 (a_1 in Table 9.4). All color models agreed with this description, which provides a linear relation between the Munsell hue coordinate and the value of the color point's azimuthal angle in the spherical model. The same result was demonstrated in the work of Indow (1980), in which the locations of Munsell colors in such chromatic planes were analyzed in detail.

Brightness. A relation between Munsell brightness value (V) and spherical angles (a_3 in Table 9.4) of color points on the achromatic plane x_3x_4 is not

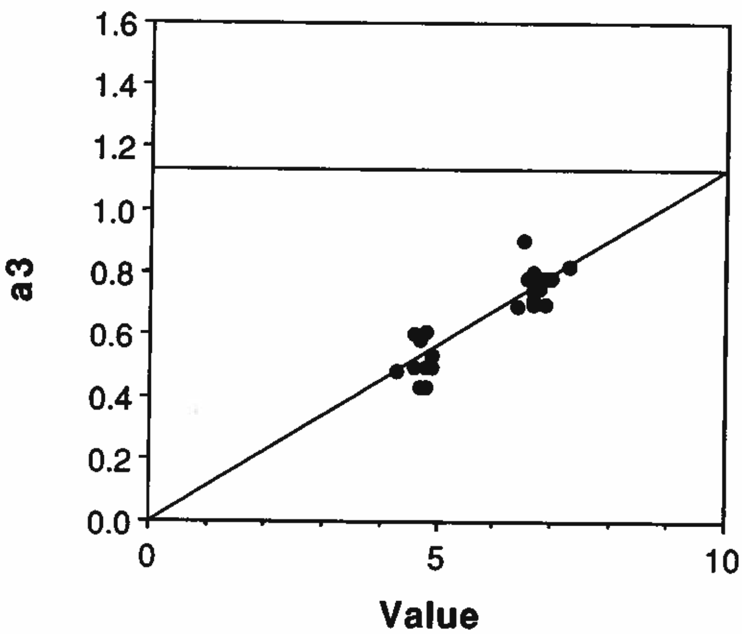

FIG. 9.4. Scatter diagram representing the relation between Munsell's scale of brightness (value) and the spherical coordinate a_3 obtained for 24 Munsell colors.

immediately evident. The situation is clarified by representing value and spherical angle in a scatter diagram (see Figure 9.4). Two clusters of points on this diagram mark two levels of brightness of the 24 Munsell colors (brightness values *V*5 and *V*7) (Indow & Kanazawa, 1960). Figure 9.4 shows a straight line drawn between the centers of the clusters and the origin of the scatter diagram: a linear model relates the two forms of brightness. Values of Munsell colors of reflected lights (abscissa) are related linearly to the brightnesses of self-luminous lights (ordinate). From this relationship, it follows that the maximal value of Munsell colors (10.0) has the intermediate value 1.1 rad on the scale of brightness of self-luminous colors.

Saturation. A scatter diagram for saturation is shown in Figure 9.5. The abscissa and the ordinate of the diagram represent Munsell chroma and the spherical coordinate a_2 for saturation, respectively, for the 24 Munsell colors. The linear relation between chroma and a_2 is very evident. The direct line that best approximates points on the scatter diagram intersects the vertical axis at a distance of 0.13 rad from the origin. This distance corresponds to nearly two JNDs in terms of the spherical model of color discrimination (Izmailov, 1980;

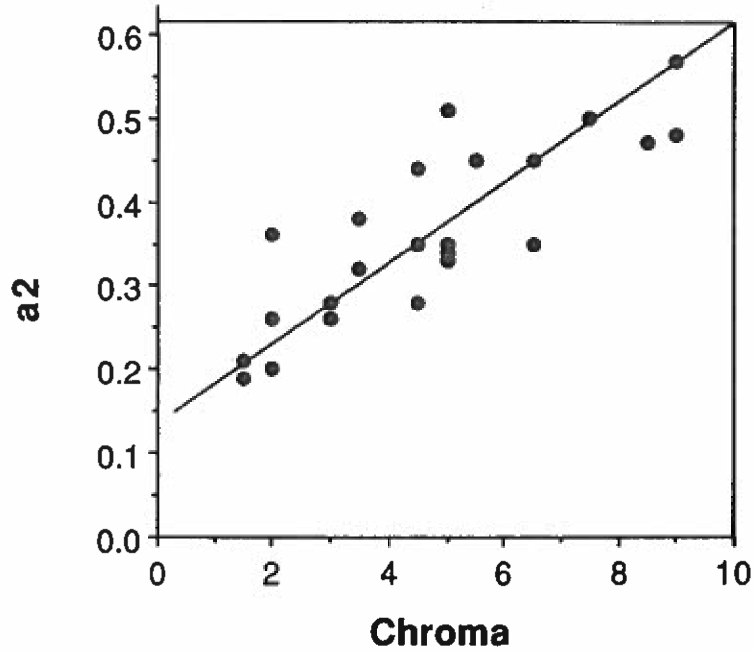

FIG. 9.5. Scatter diagram representing the relation between Munsell's scale of saturation (chroma) and the spherical coordinate a_2 obtained for 24 Munsell colors.

Sokolov & Izmailov, 1983). This small, constant shift of saturation may be due to experimental noise for a given set of stimuli evoked by influence of background and illuminance conditions.

CONCLUSION

These results represent the first step of analysis of the interrelation between the Munsell color system and the spherical model of color vision. They are far from complete. But even these results lead to positive conclusions about a color space that embeds self-luminous (aperture) colors as well as surface colors.

First, a general color space can be constructed for aperture and surface colors, which vary in hue, saturation, and brightness, by representing the three basic variables as three spherical angles of color points in this space. Euclidean distances between points in such a space coincides well with perceived color differences, both for measurements along separate color characteristic axes and for measurements between points that lie off the axes. The color space presents a solution to a problem with the cylindrical Munsell color solid, which, although the solid shows a good uniformity for each separate color variable, possesses substantial nonuniformity in interpoint distances when these points are allowed to vary in two *or more* color variables (Indow, 1980; Wyszecki & Stiles, 1982). Representing the Munsell variables as spherical coordinates provides total uniformity of color differences.

Second, relations between Munsell color characteristics and spherical angles are linear in first approximation. In this approximation, the set of surface colors does not fill the entire color space; rather it is a subset with the following limits. The hue scale for surface colors coincides with the azimuthal coordinate for aperture colors which subtends 360° (or 2π rad). Chroma has a range of 10 steps and is limited by a maximum of 0.5 rad (Figure 9.5). Value also has a range of 10 steps and is limited in the spherical model to 1.1 rad—the greater part of the spherical coordinate range (Figure 9.4). It may be of interest to note that, when represented in terms of the spherical model, one step in chroma is equivalent to two steps in value.

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